Downloaded from rsta.royalsocietypublishing.org

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Mathieu Functions of General Order: Connection Formulae, Base Functions and Asymptotic Formulae: III. The Liouville-Green Method and its Extensions

W. Barrett

Phil. Trans. R. Soc. Lond. A 1981 **301**, 99-114
doi: 10.1098/rsta.1981.0100

Email alerting service Receive free email alerts when new articles cite this article - sign up in the box at the top right-hand corner of the article or click **here**

To subscribe to *Phil. Trans. R. Soc. Lond. A* go to: **http://rsta.royalsocietypublishing.org/subscriptions**

This journal is © 1981 The Royal Society

Downloaded from rsta.royalsocietypublishing.org

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

Phil. Trans. R. Soc. Lond. A **301**, 99–114 (1981) [99]
Printed in Great Britain

MATHIEU FUNCTIONS OF GENERAL ORDER: CONNECTION FORMULAE, BASE FUNCTIONS AND ASYMPTOTIC FORMULAE

III. THE LIOUVILLE–GREEN METHOD AND ITS EXTENSIONS

By W. BARRETT
*School of Mathematics, The University of Leeds, Leeds LS*2 9*JT, U.K.*

(*Communicated by F. Ursell, F.R.S. – Received* 1 *February* 1979 – *Revised* 14 *July* 1980)

CONTENTS

An account is given of the Liouville–Green method for the approximate solution, with error estimates, of linear second-order differential equations, together with certain extensions of the method. The purpose is to make readily available a range of techniques for use in the two final parts of the present series.

The topics treated include:

(*a*) the construction of approximations in terms of both elementary and higher transcendental functions,

(*b*) the relations between approximations of the same solution in terms of different functions,

(*c*) the identification of solutions and the estimation of connection coefficients,

(*d*) uniform estimation of the error-control function in problems with more than one widely ranging parameter,

(*e*) the construction of majorants for approximating functions,

the last two being required for the derivation of satisfactory error estimates.

There is little in this part that is new, though a method of constructing approximations in terms of Bessel functions is developed specifically for application to the Mathieu equation. Apart from this, some aspects of the presentation are thought to be novel.

The Royal Society is collaborating with JSTOR to digitize, preserve, and extend access to
Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences.
www.jstor.org

Downloaded from rsta.royalsocietypublishing.org

1. General description

The purpose of these methods is to obtain approximate solutions of second-order homogeneous linear differential equations. The basic tool required is the class of Liouville transformations of pairs of variables:

$$(z, y) \to (\zeta, w),$$

where ζ is a function of z and $w = (\mathrm{d}\zeta/\mathrm{d}z)^{\frac{1}{2}}\, y$. The variables may be real or complex and ζ is required to be twice differentiable with non-vanishing first derivative. The domains of z and ζ will be intervals in the real-variable case; if the variables are complex the domains may be Riemann surfaces rather than regions in the complex plane.

Liouville transformations are characterized by the property that, when applied to a differential equation of the form

$$\mathrm{d}^2y/\mathrm{d}z^2 = F(z)\, y,$$

the transformed equation has the same form. It is in fact

$$\mathrm{d}^2w/\mathrm{d}\zeta^2 = (\mathrm{d}\zeta/\mathrm{d}z)^{-2}\, [F(z) + \tfrac{1}{2}\{\zeta, z\}]\, w,$$

where $\{\zeta, z\}$ denotes the Schwarzian derivative.

Two general properties are of significance in the applications. The first follows immediately from this characterization: the inverse of a Liouville transformation and the composition of two Liouville transformations are clearly Liouville transformations wherever they are defined. The second follows from the above formula; it is that a specified Liouville transformation applied to a general differential equation of the given type effects a non-homogeneous linear transformation on its coefficient, the coefficients in this transformation being, of course, functions.

A differential equation of the form

$$\mathrm{d}^2y/\mathrm{d}z^2 = \{u^2 f(z) + g(z)\}\, y, \tag{1.1}$$

with independent variable z, u being a parameter, for which approximate solutions are sought, is first reduced by means of a Liouville transformation as specified in (1.4*a*, *b*) below to the similar form

$$\mathrm{d}^2w/\mathrm{d}\zeta^2 = \{[u^2 + \psi(\zeta)]\, \phi(\zeta) + \chi(\zeta)\}\, w, \tag{1.2}$$

ζ being the new independent variable, where the solutions of the equation

$$\mathrm{d}^2w/\mathrm{d}\zeta^2 = \{u^2\phi(\zeta) + \chi(\zeta)\}\, w, \tag{1.3}$$

referred to as the 'basic equation', are expressible in terms of some standard functions whose properties are taken to be known, and the term $\psi(\zeta)$ is treated as a small perturbation. Solutions of the basic equation will be called 'basic functions'; the term $\chi(\zeta)$ in fact appears in only two of the basic equations discussed below.

The transformed variables (ζ, w) satisfy

$$\mathrm{d}\zeta/\mathrm{d}z = [f(z)/\phi(\zeta)]^{\frac{1}{2}}, \tag{1.4a}$$

$$[f(z)]^{\frac{1}{4}}\, y = [\phi(\zeta)]^{\frac{1}{4}}\, w. \tag{1.4b}$$

Let $Y(\zeta)$ be a solution of (1.3); then the solution of (1.2) having the same value and derivative as $Y(\zeta)$ at some prescribed point ζ_0 can be expressed by the method of variation of parameters

Downloaded from rsta.royalsocietypublishing.org

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

as the solution of a certain integral equation. In this way it is possible to show that under suitable conditions, on a certain interval or region,

$$w(\zeta) = Y(\zeta) + \mathbf{Y}(\zeta)\,\eta, \tag{1.5}$$

where
$$|\eta| \leqslant \exp\left\{c|u|^{-1} \operatorname*{var}_{[\zeta_0,\,\zeta]} \int \psi(t)\,[\phi(t)]^{\frac{1}{2}}\,dt\right\} - 1, \tag{1.5a}$$

c being a positive constant and $\mathbf{Y}(\zeta)$ a positive real-valued function not depending on the perturbation function $\psi(\zeta)$, whose behaviour is closely related to that of $|Y(\zeta)|$; a similar formula holds for $dw/d\zeta$. The symbol 'var' denotes the 'variational operator' (see Olver 1974, ch. 1, §11).

It is important to notice that this formula is valid for all non-zero values of u, not only for large values; if u is complex the domain of validity will, however, depend on arg u. The constant c may in some cases depend on u, but is bounded if u^{-1} is bounded.

If the variable is real, the variation is to be calculated over the interval with end points ζ_0, ζ, while if the variable is complex, it is to be calculated along a path joining these points and belonging to an appropriate class to be defined later. Similarly, again under suitable conditions, there is a solution of (1.2) characterized by the property that it is asymptotically equal to $Y(\zeta)$ as $\zeta \to \infty$ along a path γ_0 which may be described as 'originating from infinity', and formula (1.5) remains valid with ζ_0 replaced by $\pm\infty$ as appropriate, or conventionally by ∞ in the complex-variable case. The paths along which the variation is to be calculated will share a common (infinite) initial arc with γ_0; but see §4.1.

A simpler form of estimate for η in (1.5), which follows from (1.5a), is

$$\eta = |u|^{-1} \operatorname{var} \Psi(t)\, O(1), \tag{1.5b}$$

uniformly provided u^{-1} var $\Psi(t)$ is uniformly bounded, where

$$\Psi(t) = \int \psi(t)\,[\phi(t)]^{\frac{1}{2}}\,dt.$$

A technique for estimating var $\Psi(t)$ is outlined in §5 below.

The appropriate function $\mathbf{Y}(\zeta)$ has the property that $Y(\zeta) = \mathbf{Y}(\zeta)\, O(1)$ uniformly and the reciprocal relation is also satisfied except near zeros of $Y(\zeta)$. The remainder term in (1.5) can thus be directly compared with the principal term; the estimate (1.5a) is normally realistic in the sense that the overestimation factor is bounded except near zeros of $Y(\zeta)$.

In applying these formulae there is no difficulty in principle, though there may be considerable difficulty in practice, in taking account of the dependence of the functions $f(z)$, $g(z)$ in (1.1) on any other parameters. Problems of this kind are not treated in this part III; they are treated *ad hoc* as they arise in parts IV and V.

1.1. *Some basic equations*

(a) In (Olver 1974) three basic equations are considered, with both real and complex variables. These are described here.

(i) The basic equation $d^2w/d\zeta^2 = \pm u^2 w$ has solutions that are exponential, hyperbolic or circular functions. This equation corresponds to the L.–G. method proper, and is applicable on an interval or region in which the function $f(z)$ in (1.1) is free from zeros and singularities.

Downloaded from rsta.royalsocietypublishing.org

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

For notational compatibility when making comparison with other basic equations, the variables (ζ, w) in this equation will be renamed (ξ, v) so that the equation becomes

$$\mathrm{d}^2v/\mathrm{d}\xi^2 = \pm u^2 v, \tag{1.6}$$

and the corresponding equation (1.2) is

$$\mathrm{d}^2v/\mathrm{d}\xi^2 = \{\pm u^2 + \psi(\xi)\}\, v. \tag{1.6a}$$

With the positive sign in (1.6), either of $\mathrm{e}^{\pm u\xi}$ may be taken as basic function in (1.5); the condition to be satisfied by the path of integration in (1.5*a*) is then that it should be, in the respective cases, $\pm u\xi$-progressive, that is, that Re $(\pm u\xi)$ should be monotone non-decreasing on the path; a path on which Re $(u\xi)$ is constant is of course of both types. It may be convenient to choose an odd or even basic function, when $\xi = 0$ will be chosen as initial point and paths of either type are suitable, but to give estimates on different subdomains.

For the basic equation (1.6) the formula (1.4*a*) becomes $\mathrm{d}\xi/\mathrm{d}z = [f(z)]^{\frac{1}{2}}$, so that the formula (1.5) gives for the corresponding solution of (1.1):

$$y(z) = [f(z)]^{-\frac{1}{4}}\,\{Y(\xi) + \mathbf{Y}(\xi)\,\eta\}.$$

In a region of the z-plane containing zeros or singularities of $f(z)$, ξ is a many-valued function of z; with given initial point z_0, the corresponding value ξ_0 being either finite or at infinity, this formula with the estimate (1.5*a*) is valid on the region accessible from z_0 by $\pm u\xi$-progressive paths, as appropriate, which do not pass through such points, often called transition points of the differential equation. The Mathieu equation is used later to illustrate this.

(ii) The basic equation

$$\mathrm{d}^2w/\mathrm{d}\zeta^2 = u^2\zeta w \tag{1.7}$$

has solutions that are expressible in terms of Airy functions, a pair of independent solutions being

$$w = \mathrm{Ai}\,(u^{\frac{2}{3}}\zeta), \quad \mathrm{Bi}\,(u^{\frac{2}{3}}\zeta).$$

This basic equation is used to derive approximations that remain valid at and in the neighbourhood of a point z_0 where $f(z)$ has a simple zero; the variable ζ is that solution of (1.4*a*) for which $\zeta = 0$ at z_0. The resulting transformation is regular at z_0, so that, in particular, if $g(z)$ is analytic at z_0, then $\zeta\psi(\zeta)$ is analytic at $\zeta = 0$, as are the solutions of both (1.2) and (1.3).

By defining $\xi = \frac{2}{3}\zeta^{\frac{3}{2}}$, the appropriate paths are again $\pm u\xi$-progressive, but may pass through z_0; however, the domain of validity of (1.5) is significantly modified. By taking $Y(\zeta) = \mathrm{Ai}\,(u^{\frac{2}{3}}\zeta)$ with an arbitrary determination of $u^{\frac{2}{3}}$, and initial point such that $\arg\,(u^{\frac{2}{3}}\zeta) = 0$, the frontier of the domain in the z-plane as described under (i) above includes the two-sided arc on which $\arg\,(u^{\frac{2}{3}}\zeta) = \pi$; on this cut, ξ may be determined by continuation in either sense around $\zeta = 0$, and points on the cut are accessible by $-u\xi$-progressive paths passing through $\zeta = 0$ and then following the cut. Such paths are admissible, and the estimate (1.5*a*) remains bounded and realistic on and in the neighbourhood of the cut.

(iii) The basic equation

$$\mathrm{d}^2w/\mathrm{d}\zeta^2 = \tfrac{1}{4}\{u^2\zeta^{-1} + (\nu^2-1)\,\zeta^{-2}\}\,w \tag{1.8}$$

has solutions $\zeta^{\frac{1}{2}}\mathscr{Z}_\nu(u\zeta^{\frac{1}{2}})$, where $\mathscr{Z}_\nu$ denotes a modified Bessel function of order ν. This basic equation is used in the same way as (1.7), but when $f(z)$ has a simple pole at z_0. It is not useful in the context of Mathieu functions, but it may be remarked that, defining $\xi = \zeta^{\frac{1}{2}}$ gives appropriate paths which are again $\pm u\xi$-progressive, but with a further condition.

Downloaded from rsta.royalsocietypublishing.org

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A

For each of these three basic equations, Olver (1974) gives a construction for asymptotic series in descending integer powers of the parameter u, with remainder estimates when the series are truncated which are similar in form to (1.5a); for equation (1.6) the series is the well known Horn–Jeffreys series. If only the principal term of the series is taken, the remainder estimate is precisely equivalent to (1.5a), though the notation here differs somewhat from Olver's.

(b) Two other basic equations are required for application to the Mathieu equation. The first of these is studied by Olver (1975), but in the real-variable case only; asymptotic formulae with remainder estimates are obtained, but it did not prove possible to construct satisfactory asymptotic series. Olver introduces in the remainder formula a 'balancing function' which has a certain degree of arbitrariness; with a particular choice for this function, the estimate in fact has the form (1.5a) above. The basic equations in question are given in (i) and (ii) below.

(i) The first is

$$d^2w/d\zeta^2 = \pm\tfrac{1}{4}u^2(\zeta^2-\alpha)\,w, \tag{1.9}$$

where the parameter u is taken to be real and α is a second real parameter which takes small rather than large values. The solutions of (1.9) are expressible in terms of parabolic cylinder functions; for example, with the positive sign in the equation, two solutions represented in the notation introduced by Miller (1952) are

$$U(-\tfrac{1}{4}u\alpha,\, u^{\frac{1}{2}}\zeta), \quad \overline{U}(-\tfrac{1}{4}u\alpha,\, u^{\frac{1}{2}}\zeta).$$

The object of introducing the basic equation (1.9) is to derive approximations whose validity extends to the neighbourhood of a pair of simple zeros of $f(z)$ in (1.1), with positions depending on a second parameter and which may coalesce for some value of that parameter. It is therefore necessary to construct the Liouville transformation $(z, y) \to (\zeta, w)$ in accordance with (1.4a, b) in such a way that the two zeros of $\zeta^2-\alpha$ in (1.9) correspond to the two given zeros of $f(z)$. Let the latter be z_0, z_1; then it suffices to choose α so that

$$\int_{z_0}^{z_1}[f(z)]^{\frac{1}{2}}\,dz = \frac{1}{2}\int_{-\sqrt{\alpha}}^{\sqrt{\alpha}}(\zeta^2-\alpha)^{\frac{1}{2}}\,d\zeta,$$

whence

$$\alpha = \frac{4}{\pi}\int_{z_0}^{z_1}[f(z)]^{\frac{1}{2}}\,dz, \tag{1.9a}$$

and then to choose the arbitrary constant in the solution of (1.4a) appropriately. The transformation is then regular at both z_0 and z_1.

Define

$$\xi = \frac{1}{2}\int(\zeta^2-\alpha)^{\frac{1}{2}}\,d\zeta. \tag{1.9b}$$

Then in the complex-variable case, if $\alpha \neq 0$, the appropriate paths are once more $\pm u\xi$-progressive, so the description of the domain of validity of the resulting approximations (1.5) is similar to that for the equation (1.7). However, the frontier of the region in the z-plane now includes two cuts, one issuing from each of z_0, z_1, and the estimate (1.5a) remains finite and realistic on and in the neighbourhood of each.

If $\alpha = 0$, some modification is necessary. For in this case the simple zeros of $f(z)$ coalesce in a double zero and there are four principal domains (see (c) below) which meet at this point; continuation of ξ from any one of these to the opposite domain does not permit the construction

Downloaded from rsta.royalsocietypublishing.org

of progressive paths with initial point in one and terminal point in the other. It suffices, however, (cf. Olver 1978) to use paths that pass through the double zero and are $\pm u\xi$-progressive in each of the two domains, possibly with opposite signs; the signs are determined by the choice of branches of ξ. Equivalently, this case may be treated as limiting as $\alpha \to 0$ through either positive or negative values.

(ii) Finally, the basic equation

$$\mathrm{d}^2 w/\mathrm{d}\zeta^2 = \{u^2\alpha^2(1 \pm \zeta^{-2}) - \tfrac{1}{4}\zeta^{-2}\}\, w \tag{1.10}$$

provides approximations to Mathieu functions for a range of parameters in which none of the foregoing is suitable. Its solutions have the form

$$w = (u\alpha\zeta)^{\frac{1}{2}}\, \mathscr{Z}_\nu(u\alpha\zeta),$$

where $\mathscr{Z}_\nu$ denotes a modified Bessel function of order ν; $\nu = u\alpha$ if the sign in (1.10) is positive and $\nu = \mathrm{i}u\alpha$ if the sign is negative.

In the same way as for equation (1.9), the Liouville transformation is constructed so that the zeros of the factor $(1 \pm \zeta^{-2})$ in (1.10) correspond to a specified pair z_0, z_1 of simple zeros of $f(z)$, the parameter α being given by

$$\int_{z_0}^{z_1} [f(z)]^{\frac{1}{2}}\, \mathrm{d}z = \begin{cases} \mathrm{i}\pi\alpha & \text{if the sign is positive} \\ \pi\alpha & \text{if the sign is negative,} \end{cases} \tag{1.10a}$$

the integral being taken along a suitable path. It again turns out that $\pm u\xi$-progressive paths are appropriate in (1.5a), where

$$\xi = \alpha \int (1 \pm \zeta^{-2})^{\frac{1}{2}}\, \mathrm{d}\zeta. \tag{1.10b}$$

It may be remarked that whereas, with the basic equation (1.9), the inverse transformation $\zeta \to z$ will normally be regular in an extended region containing the interval $[-\alpha^{\frac{1}{2}}, \alpha^{\frac{1}{2}}]$, the corresponding property does not hold for equation (1.10), since $|\xi| \to \infty$ as $\zeta \to 0$. The transformation $\zeta \to \xi$ used here is a modification of that used by Olver (1954b) to derive asymptotic series for Bessel functions of large order.

The following remark illustrates the fact that the parameter u need not be thought of as large. In its application to the Mathieu equation, this basic equation, with the use of the estimate (1.5a), leads to approximations which are uniformly valid on a fixed unbounded region as $u \to 0$, provided that α varies with u in such a way that $(u\alpha)^{-1}$ remains bounded; see V, §4.

(c) At this point a pattern emerges, for the variable ξ introduced in connection with the basic equations (1.7)–(1.10) is in every case the same as the independent variable in equation (1.6a); in fact the Liouville transformation $(z, y) \to (\zeta, w)$ may be regarded as the composition of the fixed transformation $(z, y) \to (\xi, v)$ and the inverse of the transformation $(\zeta, w) \to (\xi, v)$ corresponding to a particular basic equation. Thus the appropriate paths, as described in either the ξ- or the z-plane are the same in every case. It follows that the domains of validity are the same also with the exception that certain cuts – or, in the case of equation (1.8), frontier arcs – are now included in the domain, and the appropriate estimate (1.5a) remains valid and realistic on and in the neighbourhood of such a cut or arc.

As an illustration, consider the Mathieu equation

$$\mathrm{d}^2 y/\mathrm{d}z^2 + (\lambda + 2h^2 \cos 2z)\, y = 0,$$

Downloaded from rsta.royalsocietypublishing.org

with $\lambda > 2h^2$, λ and h being real. The identification with (1.1) is

$$u = h,$$
$$f(z) = -(2\cos 2z + \lambda/h^2),$$
$$g(z) = 0;$$

$f(z)$ is an entire function with simple zeros at $z = (n+\frac{1}{2})\pi \pm ia$, where $\cosh 2a = \frac{1}{2}\lambda/h^2$.

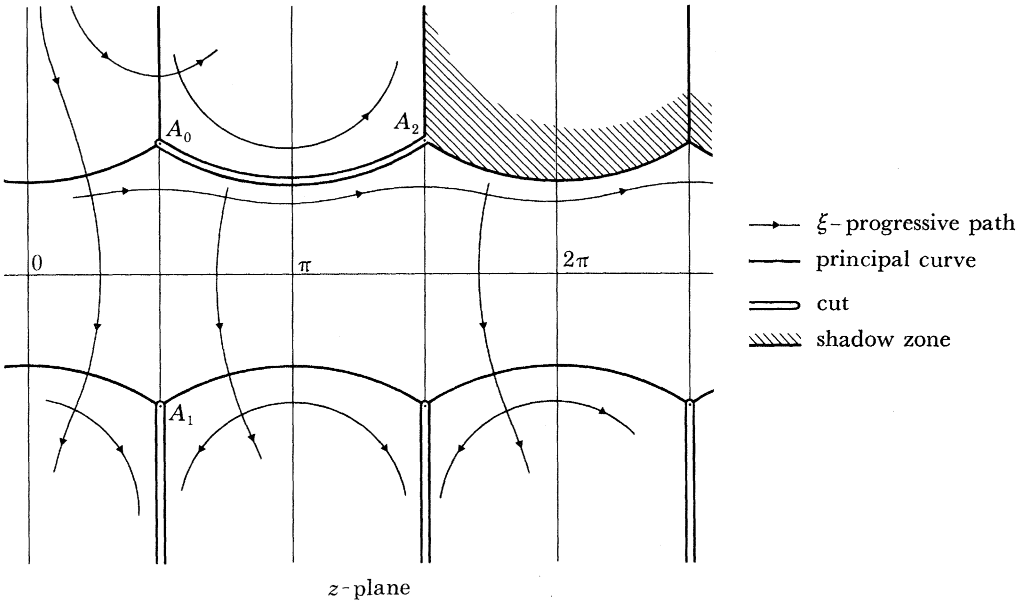

FIGURE 1. The Mathieu equation, $\lambda > 2h^2$: principal domains for the Liouville transformation.

Figure 1 shows zeros of $f(z)$ – three of them labelled A_0, A_1, A_2 –, ξ-progressive paths originating from ∞i and also level curves of Re ξ issuing from zeros of $f(z)$; at each point there are three curves meeting at equal angles. Generally, level curves of Re $(u\xi)$ issuing from transition points of the differential equation are called 'principal curves', and the regions into which they divide the z-plane are called 'principal domains' or 'Stokes regions'. Figure 1 also shows a 'shadow zone' not accessible by such progressive paths, and cuts whose removal together with the shadow zone leaves a region on which ξ has a single-valued branch, and every point of which is accessible by a progressive path lying in the region. The cuts and shadow zone (if any) depend on the choice of initial point, as also does ξ as a function of z. The principal curves and domains do not depend on $|u|$, but do depend on arg u and also on any other parameters.

Applied to the basic equation (1.6), the estimate (1.5a) is not valid on the cuts or on the shadow zone with its frontier, and becomes arbitrarily large in their neighbourhood. With equation (1.7) and with $\zeta = 0$ *either* at A_0 *or* at A_1, (1.5a) becomes valid on the corresponding cut, though not near its other extremity in the case of A_0. Equation (1.9) can be used to give approximations which remain satisfactory in the neighbourhood of both A_0 and A_1, provided that sinh $2a$ is not too large; (1.5a) is then valid on both cuts. Finally, if sinh $2a$ is not too small, equation (1.10) gives approximations which remain satisfactory in neighbourhoods of both A_0 and A_2, (1.5a) then being valid on the cut joining them and on the frontier of the shadow zone. In every case, the domain of validity is in other respects unchanged.

Downloaded from rsta.royalsocietypublishing.org

It must be emphasized that this domain is the largest on which (1.5*a*) is valid; the feasibility of estimating the variation, or the usefulness of the result – that is, whether it is sufficiently small – is quite another question, and for these reasons the necessary calculations for the Mathieu equation will be made in more restricted regions, which are nevertheless sufficiently extensive to permit the construction of a comprehensive set of asymptotic formulae.

2. The derivation of the formula (1.5)

There follows an account, without detailed proofs, of an approach to this problem which is readily applicable to the five basic equations already referred to, with either real or complex variables. The method depends on the use of suitable majorant functions for the basic functions, and also for their derivatives if approximations are required for the derivatives of solutions of (1.2). A brief description of the construction of such majorant functions appears in §6.

Let Y_1, Y_2 be two basic functions, solutions of (1.3), and suppose that $\mathbf{Y}_1(\zeta)$, $\mathbf{Y}_2(\zeta)$ are corresponding real-valued majorant functions:

$$|Y_i(\zeta)| \leqslant \mathbf{Y}_i(\zeta) \quad (i = 1, 2)$$

defined on some region; suppose also that there is a family of paths in the z-plane with common initial point z_0, whose images in the ζ-plane lie in this region and have initial point ζ_0, every path γ of the family to have the following property.

Condition A. $\mathbf{Y}_1(\zeta)/\mathbf{Y}_2(\zeta)$ *is monotone non-decreasing on* γ.

It can then be shown without difficulty, by applying standard estimation procedures to the iterative solution of the integral equation referred to above, that the following theorem holds.

Theorem 1. *The solution* $w_1(\zeta)$ *of* (1.2) *such that* $w_1(\zeta_0) = Y_1(\zeta_0)$ *and* $w_1'(\zeta_0) = Y_1'(\zeta_0)$ *satisfies*

$$w_1(\zeta) = Y_1(\zeta) + \mathbf{Y}_1(\zeta)\,\eta, \tag{2.1}$$

where

$$|\eta| \leqslant \exp\left\{\operatorname*{var}_{\Gamma}\int 2\mathscr{W}^{-1}\mathbf{Y}_1(t)\,\mathbf{Y}_2(t)\,\psi(t)\,\phi(t)\,\mathrm{d}t\right\}-1, \tag{2.1a}$$

$\mathscr{W}$ *being the constant Wronskian of* Y_1 *and* Y_2, t *a variable with the same domain as* ζ, *and* Γ *the image, with terminal point* ζ, *of an arbitrary path* γ *of the family.*

Further, if Y_1', Y_2' *have majorants* $\mathbf{Y}_1^*(\zeta)$, $\mathbf{Y}_2^*(\zeta)$ *with the property that*

$$\mathbf{Y}_1^*(\zeta)/\mathbf{Y}_2^*(\zeta) \leqslant c\mathbf{Y}_1(\zeta)/\mathbf{Y}_2(\zeta)$$

for some constant c, *then*

$$w'(\zeta) = Y_1'(\zeta) + \mathbf{Y}_1^*(\zeta)\,\eta^*, \tag{2.2}$$

where

$$|\eta^*| \leqslant \exp\left\{\operatorname*{var}_{\Gamma}\int (1+c)\,\mathscr{W}^{-1}\mathbf{Y}_1(t)\,\mathbf{Y}_2(t)\,\psi(t)\,\phi(t)\,\mathrm{d}t\right\}-1. \tag{2.2a}$$

By means of (1.4*b*), this theorem gives corresponding approximations with remainder estimates for solutions of (1.1) and for their derivatives; the formulae for the latter involve both (2.1) and (2.2). It is also readily verified that the result (2.1) with estimate (2.1*a*) is essentially invariant under Liouville transformations, that is

(*a*) the majorant functions $\mathbf{Y}_1$, $\mathbf{Y}_2$ provide naturally majorants for the corresponding solutions of any Liouville transform of (1.3), which satisfy condition A; and

(*b*) the approximations and remainder estimates for solutions of (1.1) obtained by applying

Downloaded from rsta.royalsocietypublishing.org

theorem 1 with a transformed basic equation are not affected by the transformation. This does not, however, hold for the derivatives of solutions.

For a path originating from infinity, as in §1, the integral equation defines a solution $w_1(\zeta)$ asymptotically equal to $Y_1(\zeta)$, and (2.1), (2.2) with estimates (2.1*a*), (2.2*a*) remain valid; the question whether $w_1(\zeta)$ is characterized by its asymptotic property will be examined later.

To apply theorem 1 to solutions of (1.1), a basic equation having been chosen, it is necessary to consider the choice of basic functions, the construction of majorant functions, and the resulting regions of validity in the z-plane, regions accessible from z_0 by paths on which condition A is satisfied and the variation in (2.1*a*) is finite. The equation (1.6) presents little difficulty. The natural choice of majorants for the solutions $e^{\pm u\xi}$ is $|e^{\pm u\xi}|$, and the quotient of these is monotone if and only if the path is $\pm u\xi$-progressive. By choosing these two solutions for Y_1, Y_2 in either order, the estimate (1.5*a*) follows immediately from theorem 1, since $\phi(\zeta)$ in (1.3) is here the unit constant function. The domain of validity consists of those points that are accessible from z_0 – finite or at infinity – by $\pm u\xi$-progressive paths with the appropriate sign, on which the variation in (1.5*a*) is finite. If $Y_1 = \cosh u\xi$ or $\sinh u\xi$ and $Y_2 = e^{-u\xi}$, with $\xi_0 = 0$ as initial point, then $\mathbf{Y}_1(\zeta) = \cosh \mathrm{Re}\,(u\xi)$ is suitable and (1.5*a*) is valid with $c = 2$. The paths must be $u\xi$-progressive, so that points with $\mathrm{Re}\,(u\xi) < 0$ are not accessible; this is remedied by taking $Y_2 = e^{u\xi}$.

To determine suitable solutions Y_1, Y_2 for other basic equations (1.3), consider the corresponding map

$$\zeta \to \xi : d\xi/d\zeta = [\phi(\zeta)]^{\frac{1}{2}},$$

and construct principal curves and domains in the ζ-plane corresponding to the transition points of (1.3). For each such domain on which $\mathrm{Re}\,(u\xi)$ is unbounded above, theorem 1 with (1.6) as basic equation can be used to construct a solution $Y(\zeta)$ of (1.3) that is exponentially small as $\mathrm{Re}\,(u\xi) \to \infty$ in that domain, and is asymptotically equal to $[\phi(\zeta)]^{-\frac{1}{4}}\, e^{-u\xi}$; for a principal domain on which $\mathrm{Re}\,(u\xi)$ is unbounded below, there is a similar solution with the opposite sign in the exponent. This formula also provides an estimate for $Y(\zeta)$ on the region accessible by ξ-progressive paths originating from infinity in the given principal domain. Indeed, for basic equations (1.7), (1.9), (1.10) it can be shown that $Y(\zeta)$ has a majorant satisfying (see §6 below)

$$\mathbf{Y}(\zeta) = \theta(\zeta)\,|e^{\pm u\xi}|\,O(1) \tag{2.3}$$

on this region including cuts and frontier arcs; $\theta(\zeta)$ is positive and depends on u and on any other parameter but not on the original choice of principal domain, and

$$\theta(\zeta) = |\phi(\zeta)|^{-\frac{1}{4}}\,O(1) \tag{2.3a}$$

uniformly with respect to ζ and the parameters. The situation is less simple for the equation (1.8) (Olver 1974, ch. 12).

Let $Y_1(\zeta)$, $Y_2(\zeta)$ be two such solutions; then except in certain cases the intersection of the domains of validity of the corresponding majorants (2.3) contains a set of principal domains on which the signs of the exponent in (2.3) are opposite for the two majorants and whose union forms a region which is connected and contains the initial principal domains for both Y_1 and Y_2; thus the two solutions satisfy condition A for ξ-progressive paths with appropriate sign. It can be shown further that, for given Y_1, *every* point in the ζ-plane that is accessible from infinity in the initial principal domain for Y_1 lies in this region for *some* choice of Y_2. From these

Downloaded from rsta.royalsocietypublishing.org

results, including (2.3a), it can be concluded that (1.5a) is valid on domains of the form described in §1.1(c). The extension to even and odd basic functions, which exist for equation (1.9), involves complications and will not be discussed. For an account of the topology of principal domains see Evgrafov & Fedoryuk (1966).

3. The error-control function

For application in parts IV and V it is necessary to obtain expressions for the indefinite integral in (1.5a), for each of the four basic equations (1.6), (1.7), (1.9) and (1.10); following Olver's terminology, this integral will be called the 'error-control function' (e.c.f.). In connection with the calculations that follow, the general remarks at the beginning of part III on Liouville transformations should be borne in mind.

The general formula for the (ξ, v) Liouville transform of (1.3) is

$$\mathrm{d}^2v/\mathrm{d}\xi^2 = [\phi(\zeta)]^{-1}\,[u^2\phi(\zeta)+\chi(\zeta)+\tfrac{1}{2}\{\xi,\zeta\}]\,v, \tag{3.1}$$

and a convenient formula for the Schwarzian derivative is

$$\{\xi,\zeta\} = \frac{1}{2}\frac{\mathrm{d}^2\ln\phi(\zeta)}{\mathrm{d}\zeta^2}-\frac{1}{8}\left[\frac{\mathrm{d}\ln\phi(\zeta)}{\mathrm{d}\zeta}\right]^2. \tag{3.1a}$$

Let (1.1) have (ξ, v)-transform (1.6a):

$$\mathrm{d}^2v/\mathrm{d}\xi^2 = \{u^2+\psi(z)\}\,v; \tag{3.2}$$

then the e.c.f. for the L.–G. method is

$$\int\psi(z)\,\mathrm{d}\xi. \tag{3.2a}$$

Next, transforming (1.7) to the variables (ξ, v) gives

$$\mathrm{d}^2v/\mathrm{d}\xi^2 = \{u^2-\tfrac{5}{36}(\xi-\xi_0)^{-2}\}\,v,$$

whence it follows that the application of the inverse of this transformation to (3.2) gives

$$\mathrm{d}^2w/\mathrm{d}\zeta^2 = \{u^2\zeta+\zeta\psi_1(z)\}\,w, \tag{3.3}$$

where

$$\psi_1(z) = \psi(z)+\tfrac{5}{36}(\xi-\xi_0)^{-2}, \tag{3.3a}$$

with $\psi(z)$ as in (3.2); it has already been seen that, if $g(z)$ in (1.1) is analytic where $\zeta = 0$, then the same is true of $\zeta\psi_1(z)$. Since $\phi(\zeta) = [\mathrm{d}\xi/\mathrm{d}\zeta]^2 = \zeta$, the e.c.f. is

$$\int\zeta^{\frac{1}{2}}\psi_1(z)\,\mathrm{d}\zeta = \int\psi_1(z)\,\mathrm{d}\xi. \tag{3.3b}$$

Similarly, the (ξ, v)-transform of (1.9) is

$$\mathrm{d}^2v/\mathrm{d}\xi^2 = \{\pm u^2-3(\zeta^2-\alpha)^{-2}-5\alpha(\zeta^2-\alpha)^{-3}\}\,v,$$

and applying the inverse transformation to (3.2) gives

$$\mathrm{d}^2w/\mathrm{d}\zeta^2 = \{\pm\tfrac{1}{4}u^2(\zeta^2-\alpha)+\tfrac{1}{4}(\zeta^2-\alpha)\,\psi_2(z)\}\,w, \tag{3.4}$$

where

$$\psi_2(z) = \psi(z)+3(\zeta^2-\alpha)^{-2}+5\alpha(\zeta^2-\alpha)^{-3}. \tag{3.4a}$$

Downloaded from rsta.royalsocietypublishing.org

The term $(\zeta^2-\alpha)\,\psi_2(z)$ is analytic at z_0, z_1 if $g(z)$ is analytic there, and the e.c.f. is

$$\int \psi_2(z)\,\mathrm{d}\xi. \tag{3.4b}$$

It is assumed here and for the basic equation (1.10) that α and the constant of integration in the definition of ξ are chosen in accordance with §1 (*b*).

Finally, the (ξ, v)-transform of (1.10) is

$$\mathrm{d}^2v/\mathrm{d}\xi^2 = \{u^2-\alpha^{-2}\zeta^2(\tfrac{1}{4}\zeta^2\mp 1)\,(\zeta^2\pm 1)^{-3}\}\,v,$$

and applying the inverse transformation to (3.2) gives

$$\mathrm{d}^2w/\mathrm{d}\zeta^2 = \{[u^2+\psi_3(z)]\,\alpha^2(1\pm\zeta^{-2})-\tfrac{1}{4}\zeta^{-2}\}\,w, \tag{3.5}$$

where
$$\psi_3(z) = \psi(z)+\alpha^{-2}\zeta^2(\tfrac{1}{4}\zeta^2\mp 1)\,(\zeta^2\pm 1)^{-3}. \tag{3.5a}$$

Again, the term $(1\pm\zeta^{-2})\,\psi_3(z)$ is analytic at the relevant transition points z_0, z_1 if $g(z)$ is analytic there, and the e.c.f. is

$$\int \psi_3(z)\,\mathrm{d}\xi. \tag{3.5b}$$

4. Connection coefficients; identification of solutions

Several techniques have been used to estimate connection coefficients by means of the methods under consideration. The following summarizes results that will be needed later.

(*a*) Let y_j $(j = 0, 1, 2)$ be three solutions of (1.1) and let v_j be the corresponding solutions of (1.6*a*), on some domain or arc, with prescribed branches of ξ and of $[f(z)]^{-\frac{1}{4}}$. Suppose further that

$$v_j = a_j \exp\,[(-1)^j\,u\xi]\,(1+\eta_j), \tag{4.1}$$

where the a_j are complex constants and

$$|\eta_j| \leqslant \epsilon \leqslant \tfrac{1}{8},$$

and also that there are two points in the domain or on the arc where ξ takes values ξ', ξ'' such that $|\sinh\,[2u(\xi'-\xi'')]| \geqslant \frac{1}{2}$. Then it follows by an elementary calculation that there is a connection formula

$$y_2 = \alpha_0 y_0+\alpha_1 y_1, \tag{4.2}$$

or equivalently,
$$v_2 = \alpha_0 v_0+\alpha_1 v_1, \tag{4.2a}$$

and that
$$\alpha_0 a_0/a_1 = 1+\delta_0, \tag{4.3}$$

$$\alpha_1 a_1/a_2 = \exp\,[u(\xi'+\xi'')]\,\delta_1, \tag{4.3a}$$

where $|\delta_0|, |\delta_1| \leqslant k\epsilon$, k being an absolute constant. If the domain or arc of validity of (4.1) extends to infinity in the ξ-plane, then the existence of a suitable pair of points is automatically assured. Also, if (4.1) and the corresponding formula for derivatives,

$$\mathrm{d}v_j/\mathrm{d}\xi = (-1)^j\,ua_j \exp\,[(-1)^j\,u\xi]\,(1+\eta_j'),$$

where $|\eta_j'| \leqslant \epsilon$, are valid at a single point at which $\xi = \xi'$, then (4.3), (4.3*a*) remain true, but with $\xi'+\xi''$ replaced by $2\xi'$.

Downloaded from rsta.royalsocietypublishing.org

If the parameters or the domain vary in such a way that $\eta_j \to 0$ uniformly, then (4.3) gives an asymptotic formula for α_0. No such formula is obtained for α_1; for this it is necessary to use a domain on which the roles of v_0 and v_1 are interchanged. In applying these formulae, the branches of ξ and of $[f(z)]^{-\frac{1}{4}}$ in terms of which the three approximations (4.1) will have been obtained will usually be different; the necessary substitutions must be calculated for each application.

(*b*) Sharper estimates for connection coefficients may sometimes be found by the use of a basic equation other than (1.6). Let $w_j(\zeta)$ be the solutions of (1.2) corresponding to three solutions y_j ($j = 0, 1, 2$) of (1.1), and suppose that on some domain extending to infinity,

$$w_j(\zeta) = Y_j(\zeta) + \mathbf{Y}_j(\zeta)\,\eta_j, \tag{4.4}$$

in accordance with (1.5), where $|\eta_j| \leqslant \epsilon \leqslant k_0$, the majorants having the form (2.3). If $Y_0(\zeta)$, $Y_1(\zeta)$ are independent, there is a connection formula

$$Y_2(\zeta) = a_0 Y_0(\zeta) + a_1 Y_1(\zeta).$$

Then under suitable conditions,

$$y_2(z) = \alpha_0 y_0(z) + \alpha_1 y_1(z), \tag{4.5}$$

where

$$\alpha_j = a_j(1+\delta_j) \quad (j = 0, 1) \tag{4.5a}$$

with $|\delta_0|, |\delta_1| \leqslant k\epsilon$.

It is sufficient for the purposes of this paper that (4.4) should be valid on two paths extending to infinity and satisfying the conditions of (*a*) above, one path for the estimation of each of α_0, α_1. It is however possible to formulate conditions for the suitability of a pair of paths, or indeed of a single path, simply in terms of the topology of the system of principal domains associated with the problem in hand.

4.1. *The identification of solutions*

The problem here is to identify a solution of (1.1) constructed in accordance with theorem 1 with a solution defined *either* by using some other basic equation *or* by some other means. If the initial point is finite in the ξ-plane, this can be done immediately by comparing the values of the solutions and their derivatives at the initial point. If the initial point is at infinity, then the concept usually invoked is that of recessive and dominant solutions.

For definiteness, take the path γ_0 in §1 to be a $-u\xi$-progressive path on which Re $(u\xi)$ is unbounded above and which lies in a certain principal domain. Then by using basic equation (1.6), theorem 1 determines a solution such that

$$v(\xi) \sim \mathrm{e}^{-u\xi} \tag{4.6}$$

as Re $(u\xi) \to \infty$ in any manner in this domain, subject to convergence of the e.c.f., the paths used being $-u\xi$-progressive. Any solution with this property is termed 'recessive' on the given principal domain.

On the other hand, any solution defined on the same domain by using $u\xi$-progressive paths originating in the same or a different principal domain can be shown to have the property that

$$v(\xi) \sim c\,\mathrm{e}^{u\xi}$$

as Re $(u\xi) \to \infty$ in the domain, where c is a non-zero constant. Such a solution is called 'dominant'. It is clear that the two solutions are independent and hence that the first is

Downloaded from rsta.royalsocietypublishing.org

characterized by the property (4.6). Recessive solutions may be constructed by using other basic equations; they may be identified by using asymptotic formulae for the basic function Y_1 in theorem 1 in terms of the variable ξ.

It is sometimes convenient to define a solution by means of theorem 1, $-u\xi$-progressive paths being used, but with Re $(u\xi)$ bounded above on γ_0. In this case, the argument requires some refinement, but the conclusion holds that the solution is characterized by the property (4.6) as $\xi \to \infty$ on the path γ_0. Finally, two solutions of (1.2) that are asymptotically equal to the same recessive basic function on two different paths γ originating from infinity in the same principal domain are identical provided that the e.c.f. has appropriate convergence properties; it suffices that $\psi(\zeta) = \xi^{-1-\delta}\, O(1)$ $(\delta > 0)$ uniformly as $\xi \to \infty$ in the domain. This remark serves to simplify the specification of the required family of progressive paths.

5. Estimatation of the variation of the error-control function

Theorems for this purpose which have a degree of generality, but which impose restrictions other than the essential convergence properties of the e.c.f., have limitations when applied to a problem exhibiting the wide range of configurations which arise with the Mathieu equation. The following procedure, which avoids this difficulty, was therefore used to construct the estimates required in parts IV and V.

First, an estimate for $\psi(z)$ in (1.2) is found, of the form

$$\psi(z) = \psi^*(z)\, O(1), \tag{5.1}$$

uniformly with respect to z and any parameters, where $\psi^*(z)$ is not only analytic but is such that

$$\Psi^*(z) = \int \psi^*(z)\, \mathrm{d}\xi \tag{5.1a}$$

can be found explicitly in terms of ξ or of some other convenient variable. There is then constructed a class of progressive paths γ, depending on the initial and terminal points required and on the parameters, and satisfying the conditions of the following lemma, with M constant. A proof of the lemma is given in §5.1 below.

It follows immediately from the lemma that for this class of paths,

$$\underset{\gamma}{\operatorname{var}} \left\{\int \psi(z)\, \mathrm{d}\xi\right\} = \underset{\gamma}{\operatorname{var}} \left\{\int \psi^*(z)\, \mathrm{d}\xi\right\} O(1) = \sup_{\gamma} \{|\Psi^*(z)|\}\, O(1)$$

uniformly.

Lemma 1. *Let γ be a piecewise analytic path in the z-plane with parameter τ and let Ψ^* be a function analytic on a domain containing γ.*

If
$$\underset{\gamma}{\operatorname{var}} \arg \{\mathrm{d}\Psi^*(z)/\mathrm{d}\tau\} \leqslant M, \tag{5.2}$$

then
$$\underset{\gamma}{\operatorname{var}}\, \Psi^*(z) \leqslant \sup_{\gamma} \{|\Psi^*(z)|\}\, M',$$

where M' depends only on M.

For a brief account of the variational operator as applied to real-valued functions that are not necessarily continuous (see, for example, Olver 1974, ch. 1, §11). The term 'piecewise analytic' is understood to mean that there is a finite set of compact intervals covering the

Downloaded from rsta.royalsocietypublishing.org

domain in τ of the defining function of γ such that the restriction of this function to each of the intervals is analytic, including the end points, with the proviso that, if γ extends to infinity, the corresponding subinterval in τ will be half-open (bounded or unbounded).

It may be necessary to construct different estimates (5.1) in different subdomains; the paths should then be constructed to consist of a bounded number of subarcs, each in one subdomain, and the lemma applied to each subarc separately. It should be observed that the sharpness of the estimate obtained for the variation of the e.c.f. may depend not only on the choice of estimate (5.1) but also on the choice of indefinite integral in (5.1*a*).

From the lemma there follows a corollary.

Corollary. *If, on a class of paths* γ, *the condition of lemma* 3 *is satisfied uniformly and if on each path* γ *there is a prescribed point* z^* *with the property that*

$$\sup_{\gamma} \{|\Psi^*(z)|\} = \Psi^*(z^*)\, O(1)$$

uniformly, then

$$\operatorname*{var}_{\gamma} \{\Psi^*(z)\} = \Psi^*(z^*)\, O(1)$$

uniformly.

The natural choice for z^* is the terminal point where this is possible.

5.1. *Proof of lemma* 1

Let
$$\vartheta = \arg \mathrm{d}\Psi^*(z)/\mathrm{d}\tau$$
and let
$$S = \sup |\Psi^*(z)|.$$

It is necessary to examine the nature of ϑ as a function of τ, for it is undefined not only at a junction of two analytic subarcs of γ, but also at the zeros (if any) of $\mathrm{d}\Psi^*(z)/\mathrm{d}\tau$. It is, however, not hard to show that the number of all such points is finite and that ϑ is piecewise continuously differentiable on γ, with simple jump discontinuities only.

First, however, suppose that ϑ is differentiable on γ. Then

$$\operatorname*{var}_{\gamma} \Psi^*(z) = \int_{\gamma} \left|\frac{\mathrm{d}\Psi^*(z)}{\mathrm{d}\tau}\right| \mathrm{d}\tau = \int_{\gamma} \frac{\mathrm{d}\Psi^*(z)}{\mathrm{d}\tau}\, \mathrm{e}^{-\mathrm{i}\vartheta}\, \mathrm{d}\tau$$
$$= [\Psi^*(z)\, \mathrm{e}^{-\mathrm{i}\vartheta}]_{\gamma} + \mathrm{i} \int_{\gamma} \Psi^*(z)\, \mathrm{e}^{-\mathrm{i}\vartheta} \frac{\mathrm{d}\vartheta}{\mathrm{d}\tau}\, \mathrm{d}\tau.$$

In the general case, this formula may be applied to each of the subarcs γ_j $(j = 0, 1, \ldots, n)$ on which ϑ is continuously differentiable, and the result summed.

Let z_j $(j = 1, 2, \ldots, n)$ be the junction of γ_{j-1} and γ_j, and let the two limiting values of ϑ at z_j be ϑ_j, ϑ'_j. Then $\Psi^*(z)\, \mathrm{e}^{-\mathrm{i}\vartheta}$ has a discontinuity at z_j with jump

$$\Psi^*(z_j)\, [\mathrm{e}^{-\mathrm{i}\vartheta'_j} - \mathrm{e}^{-\mathrm{i}\vartheta_j}].$$

The result of the summation may thus be expressed as

$$\operatorname*{var}_{\gamma} \Psi^*(z) = [\Psi^*(z)\, \mathrm{e}^{-\mathrm{i}\vartheta}]_{\gamma} - \sum_{j=1}^{n} \Psi^*(z_j)\, [\mathrm{e}^{-\mathrm{i}\vartheta'_j} - \mathrm{e}^{-\mathrm{i}\vartheta_j}] + \mathrm{i} \sum_{j=0}^{n} \int_{\gamma} \Psi^*(z)\, \mathrm{e}^{-\mathrm{i}\vartheta} \frac{\mathrm{d}\vartheta}{\mathrm{d}\tau}\, \mathrm{d}\tau.$$

The first term of the right-hand member does not exceed $2S$ in absolute value. Further,

$$|\mathrm{e}^{-\mathrm{i}\vartheta'_j} - \mathrm{e}^{-\mathrm{i}\vartheta_j}| \leqslant |\vartheta'_j - \vartheta_j|,$$

Downloaded from rsta.royalsocietypublishing.org

whence it follows readily that the sum of the last two terms does not exceed $S \operatorname{var}_\gamma \vartheta$ in absolute value, and so finally that

$$\operatorname{var}_\gamma \Psi^*(z) \leqslant S(2+\operatorname{var}_\gamma \vartheta).$$

Lemma 1 follows immediately; in fact we may take $M' = 2+M$.

6. Majorant functions

Olver has given majorant functions for Airy functions (Olver 1954*a*) and for parabolic cylinder functions (Olver 1960), both with complex variables. The derivation involves obtaining by the methods described above, an L.–G. approximation for the Airy function and in turn an approximation in terms of Airy functions for the parabolic cylinder functions; other basic functions may be treated similarly. However, the forms obtained involve undetermined constants; this drawback can be overcome in the following manner.

If on some region, normally unbounded, the absolute value of the quotient of the given basic function and some suitable non-vanishing analytic function is bounded, then the basic function is majorized by a multiple of the absolute value of this latter function. The appropriate factor may be found by calculating the maximum of the absolute value of the above quotient on the frontier of the region and applying the maximum principle; in practice this has been found comparatively straightforward. It does not seem feasible to construct such majorants which are sufficiently realistic both for large values of the variable and near transition points; however, the pointwise minimum of a suitable pair of such majorants has been found to be satisfactory.

For the Airy function Ai (x) the region taken is the sector $\{x\colon |\arg x| \leqslant \frac{2}{3}\pi\}$ and the provisional majorants are $c_1|x^{-\frac{1}{4}}e^{-\xi}|$ and $c_2|e^{-\xi}|$, where $\xi = \frac{2}{3}x^{\frac{3}{2}}$ takes its principal value. This leads to the formula given in V, §2.3, the majorant being of the form

$$\min\{c_1|x^{-\frac{1}{4}}|, c_2\}\,|e^{-\xi}|;$$

the majorants given for Ai (x) when $|\arg(-x)| \leqslant \frac{1}{3}\pi$ and for Bi (x) are derived from this by means of known connection formulae (see again V, §2.3).

The general form obtained for majorants for a solution of (1.3) that is asymptotically equal to $A[\phi(\zeta)]^{-\frac{1}{4}}\,e^{-u\xi}$ as Re $(u\xi) \to \infty$ may conveniently be written

$$\{A|e^{-u\xi}| + B|e^{u\xi}|\}\,\theta(\zeta), \tag{6.1}$$

a typical form for the factor $\theta(\zeta)$ being

$$\theta(\zeta) = \min\{c_1|\phi(\zeta)|^{-\frac{1}{4}}, c_2\},$$

where c_1, c_2 depend on any subsidiary parameters in the basic equation; they may, however, without significant loss of sharpness, be chosen to be independent of the region and of the particular solution of the basic equation. The second term in curly brackets in (6.1) is required only in certain regions.

The following formulae for $\theta(\zeta)$ have been obtained. The parameter u has been assumed to be real; it has been set equal to unity, this involving no further loss of generality, u being in effect a scale factor. The independent variable is denoted by x in each case.

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

Downloaded from rsta.royalsocietypublishing.org

(*a*) Basic equation (1.7); Airy functions:

$$\theta(x) = \min\{1.12|x|^{-\frac{1}{4}}, 1.29\}.$$

(*b*) Basic equation (1.9); parabolic cylinder functions, ordinary or modified, with real parameter a:

$$\theta(x) = \min\{1.2|x^2 \pm 4a|^{-\frac{1}{4}}, 0.95a^{-\frac{1}{12}}, 1.2\},$$

the upper sign referring to ordinary functions.

(*c*) Basic equation (1.10); Whittaker functions,

$$x^{\frac{1}{2}}\mathscr{Z}_\nu(x) \quad \text{or} \quad x^{\frac{1}{2}}\mathscr{Z}_{\mathrm{i}\nu}(x),$$

$\mathscr{Z}$ representing a modified Bessel function:

$$\theta(x) = \min\{1.2|x^2 \pm \nu^2|^{-\frac{1}{4}}, 1.35\nu^{-\frac{1}{3}}\},$$

provided that $\nu \geqslant 0.5$, the upper sign referring to the case of real order.

The majorants so obtained satisfy condition A (§2). Also, the formula (2.3) required to derive (1.5*a*) is valid even on a region where both terms are required in (6.1); for if ξ is defined by continuation along $-u\xi$-progressive paths, the second term never in fact significantly exceeds the first. As an illustration, this is readily verified for the majorant for Ai (x) on $\{x: |\arg(-x)| \leqslant \frac{1}{3}\pi\}$ given in IV §2.3.

The resulting estimates (1.5*a*) appear to be closely comparable in sharpness with those given by Olver (1974) for complex variables, using a more sophisticated approach to the solution of the integral equation than that proposed here, and are more readily calculable numerically.

6.1. *Application*

Suppose that an approximation for a solution of (1.1) has been found in terms of a basic function, the solution of (1.3):

$$y(z) = [f(z)/\phi(\zeta)]^{-\frac{1}{4}}\{Y(\zeta) + \mathbf{Y}(\zeta)\,\eta\}, \tag{6.2}$$

where an estimate has been obtained for η. Then η provides an estimate for the remainder relative to the principal term, except near zeros of the basic function, and this information may suffice.

Otherwise, a more convenient form for calculation is obtained by substituting (6.1) into (6.2) to give

$$y(z) = [f(z)/\phi(\zeta)]^{-\frac{1}{4}}\{Y(\zeta) + [A|\mathrm{e}^{-u\xi}| + B|\mathrm{e}^{u\xi}|]\,\theta(\zeta)\,\eta\}, \tag{6.3}$$

the factor $[f(z)/\phi(\zeta)]^{-\frac{1}{4}}$, as well as $\theta(\zeta)$, being bounded in the neighbourhood of the relevant transition points.

References (Part III)

Evgrafov, M. A. & Fedoryuk, M. V. 1966 *Russ. math. Surv.* **21**, 1–48.
Miller, J. C. P. 1952 *Proc. Camb. phil. Soc.* **48**, 428–435.
Olver, F. W. J. 1954*a* *Phil. Trans. R. Soc. Lond.* A **247**, 307–327.
Olver, F. W. J. 1954*b* *Phil. Trans. R. Soc. Lond.* A **247**, 328–368.
Olver, F. W. J. 1960 *Proc. Camb. phil. Soc.* **57**, 811–822.
Olver, F. W. J. 1974 *Asymptotics and special functions.* New York: Academic Press.
Olver, F. W. J. 1975 *Phil. Trans. R. Soc. Lond.* A **278**, 137–174.
Olver, F. W. J. 1978 *Phil. Trans. R. Soc. Lond.* A **289**, 501–548.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES